Announcements

1) HW #4 – should appear later today or tomorrow, due the week after next

2) Math Advising Session (undergraduate) Monday 11:30–1:30 CB 2076 (Math Library) – food!

3) Pi Day – tomorrow 3/14

Example 1: Every one of the floating point operations $\oplus$, $\ominus$, $\otimes$, $\oslash$, is backwards stable.

Division:

Problem: function

$$f: \mathbb{C} \times \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

$$f((x,y)) = \frac{x}{y}$$

Algorithm:

$$\tilde{f} : \mathbb{C} \times \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$$

$$\tilde{f}((x,y)) = fl(x) \oplus\!\!\!\!\div fl(y)$$

Choose some norms:

Choose $\| \ \|_\infty$ on $\mathbb{C} \times \mathbb{C} \setminus \{0\}$,

$|\cdot|$ on $\mathbb{C}$.

Recall that all norms on $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$ are equivalent, so we are free to choose whichever norm we like.

We want: some $(\tilde{x}, \tilde{y})$ "close" to (x,y) satisfying

$$\tilde{f}((x,y)) = f((\tilde{x}, \tilde{y}))$$

First, there are $\varepsilon_1, \varepsilon_2$ with $|\varepsilon_1|, |\varepsilon_2| < \varepsilon_{machine}$ such that

$$fl(x) = (1+\varepsilon_1)x$$

$$fl(y) = (1+\varepsilon_2)y$$

Also, there exists ε_3 with $|\varepsilon_3| < \varepsilon_{machine}$ and

$$fl(x) \oplus fl(y)$$

$$= (1+\varepsilon_3) \frac{fl(x)}{fl(y)}$$

Finding $\tilde{x}, \tilde{y}$

$$\tilde{f}((x,y))$$

$$= fl(x) \oslash fl(y)$$

$$= (1+\varepsilon_3)\frac{fl(x)}{fl(y)}$$

$$= (1+\varepsilon_3)\frac{(1+\varepsilon_1)x}{(1+\varepsilon_2)y}$$

$$= \frac{(1+\varepsilon_3)(1+\varepsilon_1)x}{(1+\varepsilon_2)y} = f((\tilde{x},\hat{y}))$$

where

$$\tilde{x} = (1+\varepsilon_1)(1+\varepsilon_3)x$$

$$\tilde{y} = (1+\varepsilon_2)\, y.$$

We see that $f((\tilde{x},\tilde{y})) = \tilde{f}(x,y)$,

we need to check that

(x,y) is close to $(\tilde{x},\tilde{y})$.

Check closeness

Coordinate-wise:

$$\frac{|y-\tilde{y}|}{|y|} = \frac{|y-(1+\varepsilon_2)y|}{|y|}$$

$$= |\varepsilon_2|$$

$$< \varepsilon_{machine}$$

$$\Rightarrow \frac{|y-\tilde{y}|}{|y|} = O(\varepsilon_{machine})$$

$$\frac{|x-\tilde{x}|}{|x|} = \frac{|x-(1+\varepsilon_1)(1+\varepsilon_3)x|}{x}$$

$$= |\varepsilon_1 + \varepsilon_3 + \varepsilon_1\varepsilon_3|$$

$$\le \underbrace{|\varepsilon_1| + |\varepsilon_3| + |\varepsilon_1\varepsilon_3|}_{\text{triangle inequality}}$$

$$= |\varepsilon_1| + |\varepsilon_3| + |\varepsilon_1||\varepsilon_3|$$

$$< 2\varepsilon_{machine} + \varepsilon_{machine}^2$$

$$< 3\varepsilon_{machine}$$

if $\varepsilon_{machine} < 1$.

Since we have $\varepsilon_{machine} \to 0$ for "O" notation, we get

$$\frac{|x - \tilde{x}|}{|x|} = O(\varepsilon_{machine})$$

Finally,

$$\frac{\|(x,y)-(\tilde{x},\tilde{y})\|_\infty}{\|(x,y)\|_\infty}$$

$$= \frac{\|(x-\tilde{x},\, y-\tilde{y})\|_\infty}{\|(x,y)\|_\infty}$$

$$= \frac{\max\{|x-\tilde{x}|,\, |y-\tilde{y}|\}}{\max\{|x|,|y|\}}$$

$$= O(\varepsilon) \quad \checkmark$$

Implicit Assumption in the Text

If $\| \ \|$ is a norm on a finite dimensional space, then applying the equality $\dfrac{|x_i - \tilde{x}_i|}{|x_i|} = O(\varepsilon)$

implies $\dfrac{\| x - \tilde{x} \|}{\| x \|} = O(\varepsilon)$

for $x = (x_1, x_2, \cdots, x_n)$
$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$

Sometimes we can't
do that well...

Definition: (stability)

Let $\tilde{f}: X \to Y$ be an algorithm for $f: X \to Y$.

Then $\tilde{f}$ is said to be **stable**

if for all $x \in X$, $\exists\, \tilde{x} \in X$

$$\boxed{\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(x)\|} = O(\varepsilon_{machine})}$$

and

$$\boxed{\frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{machine})}$$

Note: Every backwards stable algorithm is automatically stable since then

$$\|\tilde{f}(x) - f(\tilde{x})\|$$

$$= \| 0 \| = 0$$

$$= O(\varepsilon_{machine})$$

<u>Book</u>: "A stable algorithm gives almost the right answer to almost the right question."

Example 2: Compute $x+2$

Problem: $f: \mathbb{C} \to \mathbb{C}$

$$f(x) = x+2.$$

Algorithm: $\tilde{f}: \mathbb{C} \to \mathbb{C}$

$$\tilde{f}(x) = fl(x) \oplus 2$$

(we assume $2 \in F$)

Norm = absolute value

There exists $\varepsilon_1, \varepsilon_2$,

$|\varepsilon_1|, |\varepsilon_2| < \varepsilon_{machine}$

$fl(x) = (1+\varepsilon_1)x$

$fl(x) \oplus 2 = (1+\varepsilon_2)(fl(x)+2).$

Finding $\tilde{x}$

$$\tilde{f}(x) = fl(x) \oplus 2$$

$$= (1+\varepsilon_2)(fl(x) + 2)$$

$$= (1+\varepsilon_2)((1+\varepsilon_1)x + 2)$$

$$= (1+\varepsilon_2)(1+\varepsilon_1)x + 2\varepsilon_2 + 2$$

$$= f(\tilde{x})$$

for $\tilde{x} = (1+\varepsilon_2)(1+\varepsilon_1)x + 2\varepsilon_2$

if $\tilde{f}$ was backwards stable!

But

$$\frac{|x-\tilde{x}|}{|x|}$$

$$= \frac{|x-(1+\varepsilon_1)(1+\varepsilon_2)x - 2\varepsilon_2|}{x}$$

$$= \left| -\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2 - \frac{2\varepsilon_2}{x} \right|$$

$$\geq \frac{2\varepsilon_2}{|x|} - |\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2|$$

$\to \infty$ as $x \to 0$

So no backwards stability!

But the algorithm is stable: next class.